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Limit Sets of Cellular Automata Associated to Probability Measures

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We introduce the concept of limit set associated to a cellular automaton (CA) and a shift invariant probability measure. This is a subshift whose forbidden blocks are exactly those, whose probabilities tend to zero as time tends to infinity. We compare this probabilistic concept of limit set with the concepts of attractors, both in topological and measure-theoretic sense. We also compare this notion with that of topological limit set in different dynamical situations.

KEY WORDS: Cellular automata; limit set; attractors; Bernoulli measures.

1. INTRODUCTION

In the time evolution of many cellular automata one can observe homogeneous regions with certain simple structure separated by walls or defects which present different structure. These defects move in an apparently random manner, and disappear on collisions, so that neighboring homogeneous regions merge. When the CA starts in a random configuration, the frequency of defects steadily decreases and homogeneous regions grow in size. The dynamics of this type has been observed, e.g., in the elementary CA with rules 18, 184, 62, 54 (Boccara *et al.*[(5)]). Many computer simulations have been performed confirming this dynamics and estimating how fast the defects disappear (Grassberger[(11)]). In ref. 18 we have described this phenomenon using the Besicovitch pseudometric on the configuration space. In the present paper we adopt a probabilistic approach using the space of Borel probability measures.

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Intuitively, the system behaves as if approaching an attractor. This point of view has been adopted by Hanson and Crutchfield,[12] who describe the structure of homogeneous regions by regular languages. These regular languages in turn determine sofic subshifts, which are invariant for the CA in question. Defects can be viewed as words of minimal length not belonging to the language of the subshift. If the subshift is of finite type, this yields a finite number of types of defects. In the general case the number of type of defects is infinite.

When the CA starts in a random configuration, the probability of each defect steadily decreases to zero. This can be expressed by a topological condition in the space of Borel probability measures. The iterates of the given initial probability distribution μ approach the subspace of measures concentrated on the invariant subshift representing homogeneous regions. We call the smallest subshift with this property the μ-limit set. The forbidden blocks of this subshift are exactly those, whose probability tends to zero as time tends to infinity. Usually we assume that the initial probability measure μ is Bernoulli: letters at all positions are independent identically distributed random variables. Many results, however, can be obtained for much larger classes of measures such as Gibbs measures and Markov measures.

We show that if Σ is a topological attractor or a μ-attractor in the sense of Hurley,[13, 14] then the μ-limit set is a subset of Σ. In particular, the μ-limit set is a subset of the omega limit set. In the surjective case and when initial measures have full support we give several topological conditions, which imply that the μ-limit set is actually equal to the omega limit set: if there exist equicontinuity points, if the CA is topologically transitive and right or left permutative.

The subshifts corresponding to homogeneous regions, are usually neither attractors nor μ-attractors and we conjecture that they are μ-limit sets. This happens, e.g., in "Just gliders" CA of Milnor,[23] in which left gliders and right gliders disappear on collisions. If the initial probabilities of left and right gliders are equal, the homogenous configuration without gliders is the μ-limit set, but it is not a μ-attractor. We have shown this kind of behaviour in a different setting in ref. 18 for both "Just gliders" and 184 rules.

While we are still not able to show a similar behaviour for the elementary CA rule 18, we construct a simpler example of this type, which can be fully understood. The example implements as a CA a stochastic model studied by Erdös and Ney[8] and Adelman[1] (see also Lind[19]). Here the defects perform independent random walks with independent increments, and disappear on collisions. We show that for any Bernoulli measure μ, the μ-limit set is the homogenous configuration without defects. The result of Adelman implies that the μ-limit set is not a μ-attractor.

2. DEFINITIONS AND BACKGROUND

For a compact metric space X denote by $\mathscr{B}(X)$ the system of its Borel sets, i.e., the smallest σ-field containing the open sets. Let $\mathscr{M}(X)$ be the set of Borel probability measures defined on $\mathscr{B}(X)$. The point measure δ_x of a point $x \in X$ is given by $\delta_x(U) = 0$ if $x \notin U$ and $\delta_x(U) = 1$ if $x \in U$. The Prohorov distance between two measures μ, ν is given by

$$d_M(\mu, \nu) = \inf\{\varepsilon > 0 : \forall U \in \mathscr{B}(X), \mu(U) \leqslant \nu(B_\varepsilon(U)) \\ + \varepsilon \wedge \nu(U) \leqslant \mu(B_\varepsilon(U)) + \varepsilon\}$$

where $B_\varepsilon(U) = \{x \in X : d(x, U) < \varepsilon\}$ and $d(\cdot, \cdot)$ is the metric in X. With the Prohorov metric, the space $\mathscr{M}(X)$ of Borel probability measures is a compact metric space with the topology of weak convergence. Since for the point measures we have $d_M(\delta_x, \delta_y) = d(x, y)$, X can be viewed as a subspace of $\mathscr{M}(X)$.

The (topological) support supp(μ) of a measure $\mu \in \mathscr{M}(X)$ is the (well defined) smallest closed set with measure 1. In this paper we consider mainly measures μ which have full support, that is, for every non empty open set U, $\mu(U) > 0$. If $Y \subseteq X$ is a closed set, the space $\mathscr{M}(Y)$ can be viewed as a subspace of $\mathscr{M}(X)$ and $\mu \in \mathscr{M}(Y)$ iff supp$(\mu) \subseteq Y$.

A dynamical system is a couple (X, F), where X is a compact metric space and $F: X \to X$ is a continuous mapping. If $\mu \in \mathscr{M}(X)$ then $F\mu$ is a measure in $\mathscr{M}(X)$ defined by $F\mu(U) = \mu(F^{-1}(U))$ for every Borel set U. The map F thus extends to a map $F: \mathscr{M}(X) \to \mathscr{M}(X)$, which is continuous, so $(\mathscr{M}(X), F)$ is also a dynamical system. If $F\mu = \mu$, we say that μ is F-invariant. An F-invariant measure μ is F-ergodic, if for every F-invariant set $Y \subseteq X$ (i.e., $F(Y) \subseteq Y$) either $\mu(Y) = 0$ or $\mu(Y) = 1$. The F-invariant measure μ is mixing if for any $A, B \in \mathscr{B}(X)$ we have $\lim_{n \to \infty} \mu(A \cap F^{-n}(B)) = \mu(A)\, \mu(B)$. If μ is mixing for F then it is ergodic. We also say that μ is n-mixing if for any $A_0, \ldots, A_{n-1} \in \mathscr{B}(X)$ we have

$$\lim_{m_1, \ldots, m_{n-1} \to \infty} \mu(A_0 \cap F^{-m_1} A_1 \cap \cdots \cap F^{-m_{n-1}} A_{n-1}) = \mu(A_0) \cdot \cdots \cdot \mu(A_{n-1})$$

For a finite alphabet A denote by $|A|$ its size, by $A^* = \bigcup_{n \in \mathbb{N}} A^n$ the set of words over A, by $A^{\mathbb{N}}$ the set of one-way infinite sequences of letters from A and $A^{\mathbb{Z}}$ the set of two-way infinite sequences of letters from A. The length of a word $u = u_0 \cdots u_{n-1} \in A^n$ is denoted by $|u| = n$. The word of zero length is denoted by λ. $u \sqsubseteq v$ means that u is a subword of v, i.e., there exists k such that $v_{k+i} = u_i$ for all $i < |u|$. The subword of u starting at position i and ending at position j is denoted by $u_{[i, j]}$.

The distance of two points $x, y \in A^{\mathbb{N}}$ is

$$d(x, y) = 2^{-n} \qquad \text{where} \quad n = \min\{i \in \mathbb{N} : x_i \neq y_i\}$$

With this metric $A^{\mathbb{N}}$ is a compact metric space which is totally disconnected and perfect. The cylinder $[u]_m = \{x \in A^{\mathbb{N}} : x_{[m, m+n-1]} = u\}$ of a word $u \in A^n$ is a clopen (closed and open) set. In particular, $[\lambda]_m = A^{\mathbb{N}}$. The cylinders form a base of the topology of $A^{\mathbb{N}}$, so a Borel probability measure is determined by its value on cylinders.

In the space $A^{\mathbb{Z}}$ of doubly infinite sequences we have the metric

$$d(x, y) = 2^{-n} \qquad \text{where} \quad n = \min\{i \geqslant 0 : x_i \neq y_i \text{ or } x_{-i} \neq y_{-i}\}$$

The cylinder $[u]_l = \{x \in A^{\mathbb{Z}} : x_{[l, l+n-1]} = u\}$ of a word $u \in A^n$ starting at position $l \in \mathbb{Z}$ is a clopen set. On point measures, d_M is equivalent to d.

In this paper we consider mainly two classes of mixing measures in $\mathscr{M}(A^{\mathbb{Z}})$: Bernoulli measures and probability measures with complete connections. A Bernoulli measure on $A^{\mathbb{Z}}$ is determined by a positive probability vector $\pi = (\pi(a))_{a \in A}$, where $\pi(a) > 0$ and $\sum_{a \in A} \pi(a) = 1$. The probability of a cylinder is the product of the probabilities of its letters. If $u \in A^n$, then for every k,

$$\mu([u]_k) = \pi(u_0)\, \pi(u_1) \cdots \pi(u_{n-1})$$

A measure $\mu \in \mathscr{M}(A^{\mathbb{Z}})$ has complete connections if it satisfies: (1) $\forall u \in A^*$, $\forall w \in A^{\mathbb{N}}$, $\forall j \in \mathbb{Z}$ the limit $\mu(u \mid w) = \lim_{m \to \infty} \mu([u]_j \mid [w_{-m} \cdots w_0]_{j-(m+1)})$ exists and is positive, and (2) if for $k \geqslant 0$ we define

$$\gamma_k = \sup \left\{ \left| \frac{\mu(a \mid w)}{\mu(a \mid w')} - 1 \right| : a \in A,\, w, w' \in A^{\mathbb{N}},\, w_{[0,k]} = w'_{[0,k]} \right\}$$

then $\gamma_k \to 0$ as $k \to \infty$. We will use the following property of measures with complete connections: for any $m \geqslant 0$ there is a constant $c > 0$ such that $\mu(u \mid w) \geqslant c$ for any $u \in A^m$ and $w \in A^{\mathbb{N}}$.

A cellular automaton (CA) is a dynamical system $(A^{\mathbb{Z}}, F)$ given by

$$F(x)_i = f(x_{i-r(F)}, \ldots, x_{i+r(F)})$$

where $r(F) \geqslant 0$ is the radius and $f\colon A^{2r(F)+1} \to A$ is a local rule. A cellular automaton is continuous and commutes with the shift $\sigma\colon A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ defined by $\sigma(x)_i = x_{i+1}$.

A subshift is any subset $\Sigma \subseteq A^{\mathbb{Z}}$, which is σ-invariant (i.e., $\sigma(\Sigma) \subseteq \Sigma$), and closed. The language

$$L(\Sigma) = \{u \in A^* : \exists x \in \Sigma,\, u \sqsubseteq x\}$$

of a subshift $\Sigma \subseteq A^{\mathbb{Z}}$, is the set of words occurring in the points of Σ. Define the set of defects (minimal excluded words) of a subshift by

$$u \in D(\Sigma) \text{ iff } u \in A^* \setminus L(\Sigma) \text{ and } v \in L(\Sigma) \text{ for all proper subwords } v \text{ of } u$$

A subshift Σ is of finite type, if $D(\Sigma)$ is finite. In this case there exists a positive integer p called order, such that for $x \in A^{\mathbb{Z}}$,

$$x \in \Sigma \qquad \text{iff} \quad \forall i \in \mathbb{Z}, x_{[i, i+p-1]} \in L(\Sigma)$$

If $\Sigma \subseteq A^{\mathbb{Z}}$ is a subshift, $\mathscr{M}(\Sigma) \subseteq \mathscr{M}(A^{\mathbb{Z}})$ is the subspace of measures concentrated on Σ. We have

$$\mu \in \mathscr{M}(\Sigma) \Leftrightarrow \mu(\Sigma) = 1 \Leftrightarrow \forall k \in \mathbb{Z}, \forall u \notin L(\Sigma), \mu([u]_k) = 0$$

Since CA commute with the shift it is natural to suppose that the initial probability measure is σ-invariant. Every Bernoulli measure is σ-invariant. If μ is σ-invariant and F is a CA, then $F\mu$ is also σ-invariant. If μ is a σ-invariant measure, then for every $u \in A^*$ and every $k, l \in \mathbb{Z}$, $\mu([u]_k) = \mu([u]_l)$, so we write $\mu([u])$ omitting the subscript. We say that a point $x \in A^{\mathbb{Z}}$ is generic for a σ-invariant measure μ, if for every $u \in A^*$, the density of u in x is exactly $\mu([u])$, i.e., if,

$$\lim_{n \to \infty} \frac{\#\{i \in [-n, n] : x_{[i, i+|u|-1]} = u\}}{2n+1} = \mu([u])$$

If μ is ergodic, the set of μ-generic points has μ-measure 1. For a σ-invariant measure μ we have $\mu \in \mathscr{M}(\Sigma)$ if and only if $\mu([u]) = 0$ for all $u \in D(\Sigma)$. Similarly, a sequence of measures converges to $\mathscr{M}(\Sigma)$ if and only if the measure of every defect converges to zero.

3. ATTRACTORS AND LIMIT SETS

We recall the definitions and some properties of attractors and μ-attractors from Hurley.[13, 14] We apply it, however only to subshifts, and introduce a third, weaker concept of μ-limit set.

Definition 1. Let $(A^{\mathbb{Z}}, F)$ be a CA, $\Sigma \subseteq A^{\mathbb{Z}}$ an F-invariant subshift and μ a σ-invariant probability measure on $A^{\mathbb{Z}}$. Then

1. Σ is an attractor if there exists an F-invariant clopen set $V \subseteq A^{\mathbb{Z}}$ with

$$\Sigma = \Lambda(V) = \bigcap_{n \geq 0} F^n(V)$$

2. Σ is μ-attractor, if

$$\mu\{x \in A^{\mathbb{Z}} : \lim_{n\to\infty} d(F^n(x), \Sigma) = 0\} > 0$$

3. $\Sigma = \Lambda_\mu(F)$ is the μ-limit of F, if for every $u \in A^*$,

$$u \notin L(\Sigma) \Leftrightarrow \lim_{n\to\infty} F^n\mu([u]_0) = 0$$

Clearly, the μ-limit $\Lambda_\mu(F)$ is a well defined subshift. The omega limit set is the largest attractor

$$\Lambda(F) = \bigcap_{n\geqslant 0} F^n(A^{\mathbb{Z}})$$

If Σ is an attractor, it has a clopen invariant neighbourhood, $V \supseteq \Sigma$, whose forward images tend to Σ, so all the configurations from V are attracted to Σ. This means that defects appear only finitely many times in every central interval $[-k, k]$. Since $\mu(V)$ is positive, every attractor is μ-attractor (see Hurley[13]). If μ is σ-ergodic, then $\{x \in A^{\mathbb{Z}} : \lim_{n\to\infty} d(F^n(x), \Sigma) = 0\}$ is σ-invariant, so its measure is either 0 or 1. Thus a subshift which is μ-attractor attracts actually a set of full measure. On the other hand, the μ-limit set does not need to attract any configuration at all. In every configuration, the probability (or density) of defects decreases to zero as time goes to infinity, but the defects keep visiting any interval $[-k, k]$.

Proposition 1. If $(A^{\mathbb{Z}}, F)$ is a CA, $\mu \in \mathscr{M}(A^{\mathbb{Z}})$ a σ-ergodic measure, and $\Sigma \subseteq A^{\mathbb{Z}}$ a μ-attractor, then $\Lambda_\mu(F) \subseteq \Sigma$. In particular, $\Lambda_\mu(F) \subseteq \Lambda(F)$.

Proof. By the assumption $\mu\{x \in A^{\mathbb{Z}} : \forall\varepsilon, \exists n_0, \forall n \geqslant n_0, d(F^n(x), \Sigma) < \varepsilon\} > 0$. Since this is a σ-invariant set, its measure is 1. Given $u \notin L(\Sigma)$ with $|u| = 2m+1$, then for $\varepsilon = 2^{-m}$ we have $\mu\{x \in A^{\mathbb{Z}} : \exists n_0, \forall n \geqslant n_0, d(F^n(x), \Sigma) < 2^{-m}\} = 1$ and $\mu\{x \in A^{\mathbb{Z}} : \exists n_0, \forall n \geqslant n_0, F^n(x)_{[-m,m]} \neq u\} = 1$. Thus for every $\delta > 0$ there exists n_0 such that for every $n \geqslant n_0$,

$$F^n\mu([u]_{-m}) = \mu\{x \in A^{\mathbb{Z}} : F^n(x)_{[-m,m]} = u\} < \delta$$

so $\lim_{n\to\infty} F^n\mu([u]_{-m}) = 0$. ∎

The proof of the next proposition is left to the reader.

Proposition 2. If $\lim_{n\to\infty} F^n\mu = \nu$ in $\mathscr{M}(X)$, then $\Lambda_\mu(F) = \mathrm{supp}(\nu)$.

The following examples show that the inclusions between the three concepts of attraction are strict. We denote by $\mathbf{2}=\{0,1\}$ and for $a \in A$ we denote by a^∞ the shift periodic point in $A^{\mathbb{Z}}$, all coordinates of which are a.

Example 1. $A=\mathbf{2}$, $F(x)_i = x_{i-1}x_ix_{i+1}$.

The subshift $\{0^\infty\}$ is an attractor, since the cylinder $V=[0]_0=\{x : x_0=0\}$ is invariant and $\omega(V)=\{0^\infty\}$. The omega limit is larger,

$$\Lambda(F)=\{x \in \mathbf{2}^{\mathbb{Z}} : \forall n>0,\ 10^n1 \not\sqsubseteq x\}$$

so it contains another fixed point 1^∞. For every σ-invariant measure μ, such that $\text{supp}(\mu) \neq \{1^\infty\}$, we have $\Lambda_\mu(F)=\{0^\infty\} \neq \Lambda(F)$.

Example 2. $A=\mathbf{2}$, $F(x)_i = x_{i+1}x_{i+2}$, for $x \in A^{\mathbb{Z}}$, $i \in \mathbb{Z}$.

The fixed point 0^∞ is not an attractor, so the only attractor is $\Lambda(F)$ which is the same as in Example 1. Nevertheless, for any σ-ergodic measure μ, (in particular for every Bernoulli measure), $\{0^\infty\}$ is a μ-attractor, so $\Lambda_\mu(F)=\{0^\infty\} \neq \Lambda(F)$.

In the next example we consider some random walks and Markov chains. Let I be a countable set of states. A sequence $(X_i)_{i \geqslant 0}$ of I-valued random variables is a stationary Markov chain, if

$$\Pr[X_{n+1}=j \mid X_n=i, X_{n-1}=i_{n-1},..., X_0=i_0]=\Pr[X_{n+1}=j \mid X_n=i]=P_{ij}$$

$P^n_{ij}=\sum_{k \in I} P^{n-1}_{ik} P_{kj}$ is the probability of transition from i to j in n steps. A Markov chain is aperiodic and irreducible, if there exists n_0 such that for every $i, j \in I$, for all $n \geqslant n_0$, $P^n_{ij}>0$. A state $i \in I$ is recurrent, if $\Pr[\exists n>0, X_n=i \mid X_0=i]=1$ and this happens iff $\sum_{n \geqslant 0} P^n_{ii}=\infty$. A $\mathbb{Z}^n$-valued Markov chain is a random walk, if $(X_{n+1}-X_n)_{n \geqslant 0}$ are independent, identically distributed random variables. An irreducible and aperiodic random walk in $\mathbb{Z}$ and $\mathbb{Z}^2$ is recurrent iff $E(X_{n+1}-X_n)=0$ (see Karlin[15]).

Example 3 (Just Gliders). $A=\{-1,0,1\}$, for $x \in A^{\mathbb{Z}}$, $i \in \mathbb{Z}$,

$$\begin{aligned}
F(x)_i &= 1 && \text{if} \quad x_{i-1}=1, \quad x_i \neq -1, \quad x_{i+1} \neq -1 \\
F(x)_i &= -1 && \text{if} \quad x_{i+1}=-1, \quad x_i \neq 1, \quad x_{i-1} \neq 1 \\
F(x)_i &= 0 && \text{otherwise}
\end{aligned}$$

Thus 1 is a particle (glider) which moves to the right with velocity 1 and -1 moves to the left with velocity -1. When two particles cross, they both annihilate. Gilman[10] shows, that if μ is a Bernoulli measure and $\mu([1])>\mu([-1])$, then $\mathbf{2}^{\mathbb{Z}}$ is a μ-attractor. For any Bernoulli measure with $\mu([1])=\mu([-1])$, we have shown in ref. 18, that $\{0^\infty\}$ is not a μ-attractor but the probability of defects decreases to zero, i.e., $\Lambda_\mu(F)=\{0^\infty\}$.

Example 4 (Random Walk). $A=\mathbf{2}\times\mathbf{2}\times\mathbf{2}$,

$$
\begin{aligned}
F(x, y, z)_i=(x_{i-2}, y_{i+2}, 1)& \\
&\text{if}\quad z_{i+1}=1,\ x_{i+1}+y_{i+1}=0,\ z_i\cdot(x_i+y_i)=0,\\
&\qquad\ z_{i-1}\cdot(x_{i-1}+y_{i-1})=0\\
&\text{or}\quad z_{i-1}=1,\ x_{i-1}+y_{i-1}=1,\ z_i\cdot(x_i+y_i+1)=0,\\
&\qquad\ z_{i+1}\cdot(x_{i+1}+y_{i+1}+1)=0\\
F(x, y, z)_i=(x_{i-2}, y_{i+2}, 0)&\qquad \text{otherwise}
\end{aligned}
$$

All additions are modulo 2. Denote by $\pi_i\colon A^{\mathbb{Z}}\to\mathbf{2}^{\mathbb{Z}}$, $i=1, 2, 3$, the projections in each coordinate, so $\pi_1\colon (A^{\mathbb{Z}}, F)\to(\mathbf{2}^{\mathbb{Z}}, \sigma^{-2})$, $\pi_2\colon (A^{\mathbb{Z}}, F)\to(\mathbf{2}^{\mathbb{Z}}, \sigma^{2})$ are factor maps. Ones in the third coordinate act as particles which move to the left when the sum (modulo 2) of the first two coordinates is zero, and to the right when this sum is one. When two particles intend to cross or to go to the same site, they are both annihilated. Suppose that μ is a Bernoulli measure on $A^{\mathbb{Z}}$. Since, for $i=1, 2$, $\pi_i\mu$ are Bernoulli measures which are invariant for both σ^2 and σ^{-2} respectively, every particle performs a random walk with independent increments from $\{-1, 1\}$ until it collides with a neighboring particle. In general, these random walks are not symmetric, the probability of going to the right may differ from the probability of going to the left. Nevertheless, the distance of two neighboring particles performs a symmetric random walk with absorbing states 0 and -1 (which represent annihilation)—unless one of these particles annihilates with its other neighbour. Thus the "Random walk" CA implements the behaviour which has been observed and conjectured in elementary CA with rules 18, 54, etc.

Proposition 3. Let $(A^{\mathbb{Z}}, F)$ be the "Random walk" CA from Example 4. For any Bernoulli measure μ, $\Lambda_\mu(F)=2^{\mathbb{Z}}\times 2^{\mathbb{Z}}\times\{0^\infty\}$, but $\Lambda_\mu(F)$ is not a μ-attractor.

Proof. Using generic points we show that the density of particles decreases to some limit. If the limit were positive, there would exist an invariant measure with this density of particles, but the probability of annihilation would still be positive. Define a particle of a state $(x, y, z) \in A^{\mathbb{Z}}$ as a finite or infinite sequence of integers $t = (t_0, ..., t_{N-1})$ or $t = (t_0, t_1, ...)$ respectively, such that

$$\pi_3(F^n(x, y, z))_{t_n} = 1, \qquad |t_{n+1} - t_n| = 1$$

whenever t_n and t_{n+1} are defined. Also, if a particle $t = (t_0, ..., t_{N-1})$ cannot be prolonged, we say that it is annihilated at time N. We have

$$t_{n+1} = t_n - 1 \qquad \text{if} \quad x_{t_n + 2n} + y_{t_n - 2n} = 0$$

$$t_{n+1} = t_n + 1 \qquad \text{if} \quad x_{t_n + 2n} + y_{t_n - 2n} = 1$$

Since $(t_n + 2n)_n$ is increasing and $(t_n - 2n)_n$ is decreasing, $(x_{t_n + 2n} + y_{t_n - 2n})_n$ is a sequence of independent random variables, so $(t_n)_n$ is a random walk. If $(t_n)_{n \leqslant N}$ is a finite sequence, this is a truncated random walk, which ends when a particle is annihilated. If t, s are neighboring particles and, say, $t_n < s_n$, then $x_{t_n + 2n} + y_{t_n - 2n}$, $x_{s_n + 2n} + y_{s_n - 2n}$ are independent. Thus $u_n = s_n - t_n$ is a truncated random walk in $\{-1, 0, 1, ...\}$. The length of u is the minimum of the lengths of s and t. If these lengths are not equal, one of the particles is annihilated with another particle. If the lengths of s and t are equal and finite, the two particles annihilate when u enters one of the absorbing states -1 or 0. Moreover u is symmetric:

$$\text{Prob}[u_{n+1} - u_n = 2] = \text{Prob}[u_{n+1} - u_n = -2] \qquad \text{for} \quad u_n > 0$$

We show now that the density of particles tends to zero. Denote by $Y = \{(x, y, z) \in A^{\mathbb{Z}} : z_0 = 1\}$ the cylinder of configurations containing a particle at site 0. Let (x, y, z) be a generic point for a Bernoulli measure $\mu \in \mathcal{M}(A^{\mathbb{Z}})$ and put

$$a_n = \lim_{k \to \infty} \frac{\#\{i \in [-k, k] : \sigma^i F^n(x, y, z) \in Y\}}{2k+1} = F^n \mu(Y)$$

Since $F^n(x, y, z)$ is a generic point for $F^n \mu$, and the number of particles at time $n+1$ in interval $[-k, k]$ is at most the number of particles in interval $[-k-1, k+1]$ at time n, we get

$$a_{n+1} \leqslant \lim_{k \to \infty} \frac{\#\{i \in [-k-1, k+1] : \sigma^i F^n(x, y, z) \in Y\}}{2k+1} = a_n$$

so the limit $\lim_{n\to\infty} a_n = a$ exists. Suppose that $a > 0$. Since $\mathscr{M}(A^{\mathbb{Z}})$ is a compact space, there exists an increasing sequence $(n_i)_{i\in\mathbb{N}}$ for which the Cesàro mean of μ converges, i.e., there exists a limit

$$\nu = \lim_{i\to\infty} \frac{1}{n_i} \sum_{j=0}^{n_i-1} F^j\mu$$

Then ν is σ-invariant, $F\nu = \nu$, $\nu(Y) = a$, and the projections $\pi_1\nu$, $\pi_2\nu$ are Bernoulli. Since $\nu(Y) > 0$, there exists $j > 0$ such that $\nu\{(x, y, z) \in A^{\mathbb{Z}} : z_0 = z_j = 1\} > 0$. If the particles at sites 0 and j were the only particles in the configuration, there would exist time $p > 0$, such that these particles would annihilate by time p with some positive probability ε. This is an event which depends only on the central cylinder of length $2p+1$. Using a generic point for the measure ν, we get $F^p\nu(Y) \leqslant \nu(Y) - \varepsilon$ and this is a contradiction. Thus we have proved $\lim_{n\to\infty} F^n\mu(Y) = a = 0$, so for every $u \in A^*$ which contains a particle, $\lim_{n\to\infty} F^n\mu([u]) = 0$, and $\Lambda_\mu(F) = \mathbf{2}^{\mathbb{Z}} \times \mathbf{2}^{\mathbb{Z}} \times \{0^\infty\}$. As proved by Adelman,[1] the site zero is visited by a particle with probability one and it follows that it is visited infinitely many times with probability one. This means that $\Lambda_\mu(F)$ is not a μ-attractor. ∎

4. LIMIT SETS AND DYNAMICS OF CA

In this section we state some relations between the μ-limit set, the limit set and the dynamics of one-dimensional cellular automata. We state the equality $\Lambda_\mu(F) = \Lambda(F)$ in different dynamical situations and for different shift invariant probability measures of full topological support.

We follow the classification of one dimensional cellular automata which is based on the existence of equicontinuity points.[17] We will state this classification after recalling some concepts.

Let F: $A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a CA. It is said to be (topologically) transitive if for any non empty open sets U and V in $A^{\mathbb{Z}}$ there exists $n > 0$ such that $F^{-n}(U) \cap V \neq \emptyset$. It is (topologically) mixing if $F^{-n}(U) \cap V \neq \emptyset$ for all sufficiently large n. Mixing implies transitivity and every transitive system is surjective.

The CA F has equicontinuous points if and only if there exist words called *markers*, that is: there is $w \in A^*$ such that $|w| = 2a + r(F)$ for some $a \in \mathbb{N}$, and

$$\forall j \in \mathbb{Z}, \forall x, y \in [w]_{-a+j}, \forall n \geqslant 0, F^n(x)_{[j, j+r(F)-1]} = F^n(y)_{[j, j+r(F)-1]}$$

Put $E(F)$ to be the set of equicontinuous points. In words, a marker is a block in $A^{2a+r(F)}$ for some $a \in \mathbb{N}$ such that whenever it appears in position

$j-a$ of a point of $x \in A^{\mathbb{Z}}$, then it determines the value of $F^n(x)_{[j, j+r(F)-1]}$ for any $n \in \mathbb{N}$ independently of the values in coordinates outside the marker. A proof of this fact can be found in Kůrka.[17] In the multiplication CA of Example 1, every point except the shift periodic point 1^∞ is equicontinuous, and 0 is a marker: if $x_0 = 0$, then $F^n(x)_0 = 0$ for all $n \geqslant 0$. In contrast, the CA from Example 2 does not have equicontinuous points.

A cellular automaton is sensitive if and only if it does not have equicontinuous points, and every transitive *CA* is sensitive (see Kůrka[17] for a proof). Among sensitive CA we distinguish positive expansive ones, that is, for any $x, y \in A^{\mathbb{Z}}$, $x \neq y$, there is $n \in \mathbb{N}$ such that the iterates $F^n(x)_{[-r(F), r(F)-1]} \neq F^n(y)_{[-r(F), r(F)-1]}$. The dynamics of positively expansive CA has been completely described in refs. 6, 3, and 24. The CA from Example 2 is sensitive but not positively expansive, A classical example of positively expansive CA is

Example 5 (Addition CA). $A = \mathbf{2}$, $F(x)_i = x_{i-1} + x_i + x_{i+1} \bmod 2$, for $x \in A^{\mathbb{Z}}$, $i \in \mathbb{Z}$.

Thus we classify CA in four classes: (1) $E(F) = A^{\mathbb{Z}}$, (2) $E(F) \neq \varnothing$ but it is not equal to $A^{\mathbb{Z}}$, (3) F is sensitive to initial conditions but it is not positively expansive and (4) F is positively expansive.

Fundamental classes of maps in symbolic dynamics are the family of right permutative CA and the family of left permutative CA. A CA $(A^{\mathbb{Z}}, F)$ is right permutative if and only if

$$\forall u \in A^{2r(F)}, \forall a, b \in A, a \neq b, F(ua) \neq F(ub)$$

Analogously we can define left permutative property.

We begin our study with CA having equicontinuous points. Let us remark that if F is equicontinuous ($E(F) = A^{\mathbb{Z}}$) then for any shift invariant probability measure $\mu \in \mathscr{M}(A^{\mathbb{Z}})$ of full topological support there are $p, T \in \mathbb{N}$ such that[4]

$$\lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} F^i \mu = \frac{1}{p} \sum_{i=0}^{p-1} F^i(F^T \mu)$$

Therefore we get the equality $\Lambda(F) = \Lambda_\mu(F)$. In fact, if $u \notin \Lambda_\mu(F)$ then $(1/p) \sum_{i=0}^{p-1} F^i(F^T \mu[u]_0) = 0$, which implies that $u \notin \Lambda(F)$.

Given a word $w \in A^*$ and two non-negative integers $s \leqslant m$, we define (as in ref. 4) the set

$$D(w, m, s) = \bigcup_{i=-m}^{-s} \bigcup_{j=s}^{m} ([w]_{i-|w|+1} \cap [w]_j)$$

Thus $D(w, m, s)$ consists of all configurations which contain at least one occurrence of w finishing at $[-m, -s]$ and also at least one occurrence of w starting at $[s, m]$. Clearly $D(w, m, s) \subseteq D(w, m+1, s)$. If F has equicontinuous points and w is a marker for F, then there are integers $t(w, m, s)$ and $p(w, m, s)$ such that any point $x \in D(w, m, s)$ satisfies $F^{t(w, m, s)+ip(w, m, s)}(x)_{[-s, s]} = F^{t(w, m, s)}(x)_{[-s, s]}$ for any $i \geqslant 0$. The following lemma strengthens this property for surjective cellular automata. A related result can be found in ref. 4.

Lemma 1. Let $(A^{\mathbb{Z}}, F)$ be a surjective CA having equicontinuous points and let $w \in A^*$ be a marker for F. Then any σ-periodic point $x \in A^{\mathbb{Z}}$ containing w satisfies $F^{iM}(x) = x$ for any $i \geqslant 0$ and some $M \geqslant 0$.

Proof. By hypothesis we have that $\Lambda(F) = A^{\mathbb{Z}}$. Let x be a σ-periodic point containing w. Let T be the period of x and $v = x_{[0, T-1]}$, that is, $x_{i+jT} = v_i$ for $j \in \mathbb{Z}$ and $i \in \{0,..., T-1\}$. Moreover, we assume v has the subword w. We put $C = [vvv]_{-T}$. Since F is surjective, the uniform Bernoulli measure of $A^{\mathbb{Z}}$ is invariant with respect to F (and has full support). Therefore, by using Poincaré recurrence Theorem we deduce the existence of $y \in C$ and $M > 0$ such that $F^M(y) \in C$. Finally, since v contains the marker w we get that

$$F^n(x)_{[0, T-1]} = F^n(y)_{[0, T-1]}$$

Therefore $F^M(x) = x$ because x is σ-periodic. ∎

Proposition 4. Let $(A^{\mathbb{Z}}, F)$ be a surjective CA having equicontinuous points and let $\mu \in \mathscr{M}(A^{\mathbb{Z}})$ be a σ-ergodic probability measure of full topological support. Then $\Lambda_\mu(F) = \Lambda(F)$.

Proof. Since F is surjective we have $A^{\mathbb{Z}} = \Lambda(F)$. Let u be a word in the limit set of F and let us suppose that u is not in $L(\Lambda_\mu(F))$, that is, $\lim_{n\to\infty} F^n\mu([u]_0) = 0$. Put $s = |u|$. First we show the following assertion.

Claim. For $m \geqslant s$ there is $M \geqslant 1$ such that $F^{-iM}([u]_0) \cap D(w, m, s) = [u]_0 \cap D(w, m, s)$ for any $i \geqslant 0$, where $w \in A^*$ is any fixed marker for F.

Fix $j \geqslant 0$. If $y \in F^{-j}([u]_0) \cap D(w, m, s)$ then there is a σ-periodic point $x \in F^{-j}([u]_0) \cap D(w, m, s)$ such that

$$F^k(x)_{[0, |u|-1]} = F^k(y)_{[0, |u|-1]}$$

for any $k \geqslant 0$ and whose period is less than $2m + 1 + 2\,|w|$. By Lemma 1, there is M such that $F^{kM}(x) = x$ for any $k \geqslant 0$. Therefore,

$$y \in F^{-((j \bmod M) + kM)}([u]_0) \cap D(w, m, s)$$

for any $k \geqslant 0$. Since there is a finite number of σ-periodic points of period $2m + 1 + 2\,|w|$ in $D(w, m, s)$ we can assume that M is a constant independent of the periodic point. Then we have that for any $j \geqslant 0$

$$F^{-j}([u]_0) \cap D(w, m, s) \subseteq F^{-((j \bmod M) + kM)}([u]_0) \cap D(w, m, s)$$

This inclusion proves the claim.

Since μ is ergodic for the shift and has full topological support we get that

$$\lim_{m \to \infty} \mu(D(w, m, s)) = \mu\left(\bigcup_{m \in \mathbb{N}} D(w, m, s)\right) = 1$$

and

$$F^n\mu([u]_0) = \lim_{m \to \infty} \mu(F^{-n}([u]_0) \cap D(w, m, s))$$

(the limit in m is non decreasing).

Fix $\varepsilon > 0$ and $m \geqslant s$. By our assumption, if i is large enough then

$$\mu(F^{-iM}([u]_0) \cap D(w, m, s)) \leqslant \varepsilon$$

Therefore, using the claim we deduce that $\mu([u]_0 \cap D(w, m, s)) \leqslant \varepsilon$. Taking first the limit when m tends to infinity and then the limit when ε tends to zero we conclude that $\mu([u]_0) = 0$. This is a contradiction because μ has full topological support and $u \in L(\Lambda(F)) = A^*$. ▮

The dynamics of sensitive cellular automata (class (3)) is quite unknown. In what follows we consider the class of transitive CA, therefore surjective and sensitive, which are left or right permutative. Also we consider natural measures in statistical mechanics: Gibbs measures, Markov measures and Bernoulli measures.

Proposition 5. Let $(A^{\mathbb{Z}}, F)$ be a right (resp. left) permutative CA and let $\mu \in \mathcal{M}(A^{\mathbb{Z}})$ be a shift invariant probability measure with complete connections. If μ has full topological support then $\Lambda_\mu(F) = \Lambda(F)$.

Proof. There exists $m \in \mathbb{Z}$ such that $G = F \circ \sigma^m$ is one-sided transitive, i.e., $G(x)_i = g(x_i \cdots x_{i+r(G)})$ for some local rule $g\colon A^{r(G)+1} \to A$ and some integer $r(G) > 0$. Since μ is shift invariant and F is right permutative, G is also right-permutative and $G^n\mu = F^n\mu$ for all $n \in \mathbb{N}$. Thus we can assume that F is one-sided transitive. Let $u \in A^*$. Then, for $n > 1$

$$
\begin{aligned}
F^n\mu([u]_0) &= \mu(F^{-n}([u]_0)) \\
&= \sum_{a \in k(u)} \sum_{v \in p(a, n-1)} \sum_{w \in c(v, u, n)} \mu([vw]_0) \\
&= \sum_{a \in k(u)} \sum_{v \in p(a, n-1)} \sum_{w \in c(v, u, n)} \mu([w]_0 \mid [v]_{-|v|})\, \mu([v]_0)
\end{aligned}
$$

where

$$
\begin{aligned}
k(u) &= \{a \in A^r : F^{-1}([u]_0) \cap [a]_0 \neq \varnothing\} \\
p(a, n-1) &= \{v \in A^{nr} : F^{-(n-1)}([a]_0) \cap [v]_0 \neq \varnothing\} \\
c(v, u, n) &= \{w \in A^{|u|} : F^n(vw) = u\}
\end{aligned}
$$

Since μ has complete connections there is a positive constant c, independent of w, v and the length of v such that $\mu([w]_0 \mid [v]_{-|v|}) \geqslant c$. Therefore,

$$
\begin{aligned}
F^n\mu([u]_0) &\geqslant \sum_{a \in k(u)} \sum_{v \in p(a, n-1)} \sum_{w \in c(v, u, n)} c \cdot \mu([v]_0) \\
&\geqslant c \cdot \sum_{a \in k(u)} \sum_{v \in p(a, n-1)} \mu([v]_0) \cdot \# c(v, u, n) \\
&\geqslant c \cdot \sum_{a \in k(u)} F^{n-1}\mu([a]_0) \geqslant 0
\end{aligned}
$$

In the last series of inequalities we use the fact that $\# c(v, u, n) \geqslant 1$ for any right permutative CA. We conclude that if $u \notin L(\Lambda_\mu(F))$ then for any $a \in k(u)$,

$$
\lim_{n \to \infty} F^n\mu([u]_0) = 0 \qquad \text{and} \qquad \lim_{n \to \infty} F^n\mu([a]_0) = 0
$$

Finally, since F is transitive, we get that for any $a \in A^r$, $\lim_{n \to \infty} F^n\mu([a]_0) = 0$. This is a contradiction since $\sum_{a \in A^r} F^n\mu([a]_0) = 1$ for any $n \in \mathbb{N}$. ∎

The last proposition can be strengthened for some classes of positively expansive linear CA (which in particular are right and left permutative).

Proposition 6. Let $(A, +)$ be a finite Abelian group such that $|A| = p$ for some prime number p. Let $(A^{\mathbb{Z}}, F)$ be the addition cellular automaton defined for each $x \in A^{\mathbb{Z}}$ and $i \in \mathbb{Z}$ by

$$F(x)_i = \sum_{j=0}^{r(F)} \alpha_j(x_{i+j})$$

where $\{\alpha_j : j = 0,..., r(F)\}$ is a family of commuting isomorphisms of $(A, +)$. If $\mu \in \mathcal{M}(A^{\mathbb{Z}})$ is a shift invariant probability measure $r(F)$-mixing with respect to the shift and with full topological support then $\Lambda_\mu(F) = \Lambda(F)$.

Proof. Set $r(F) = r$. Let u be a word in the limit set of F. If u is not in $\Lambda_\mu(F)$ then $\lim_{n \to \infty} F^n\mu([u]_0) = 0$. A simple computation yields to the following formula: for $x \in A^{\mathbb{Z}}$,

$$F^m(x)_0 = \sum_{i_r=0}^{m} \cdots \sum_{i_1=0}^{i_2} \binom{i_2}{i_1}_p \cdots \binom{m}{i_r}_p \alpha_0^{i_1} \circ \cdots \circ \alpha_{r-1}^{i_{r-1}-i_r} \circ \alpha_r^{m-i_r}(x_{\sum_{j=1}^r j(i_{j+1}-i_j)})$$

where $i_{r+1} = m$ and $\binom{a}{b}_p$ is the binomial coefficient modulo p. Using Lucas' formula (see ref. 20) for combinatorial numbers we deduce that for any $m \geqslant 0$ and $x \in A^{\mathbb{Z}}$, $F^{p^m}(x)_0 = \sum_{j=0}^r x_{jp^m}$. Then for m large enough we get

$$F^{p^m}\mu([u]_0) = \sum_{v_0,..., v_{r-1} \in A^{|u|}} \mu([v_0]_0 \cap [v_1]_0 \cap \cdots \cap [v_r]_{rp^m})$$

where for each $v_0,..., v_{r-1} \in A^{|u|}$ the word v_r is the unique word in $A^{|u|}$ such that $F^{p^m}(v_0 w_0 v_1 w_1 \cdots w_{r-1} v_r) = u$ for any $w_0,..., w_{r-1} \in A^{p^m - |u|}$. Then, by taking the limit when m tends to ∞ and using the r-mixing property of μ we get

$$\sum_{v_0,..., v_{r-1} \in A^{|u|}} \mu([v_0]_0) \cdot \mu([v_1]_0) \cdot \cdots \cdot \mu([v_r]_0) = 0$$

which contradicts the fact that μ has full topological support. ∎

To finish let us observe that since the measures μ we are considering are invariant for the shift map then for any $m \in \mathbb{Z}$ we have $\Lambda_\mu(F) = \Lambda_\mu(F \circ \sigma^m)$. Then all of our results apply to all the CA considered in this paper composed with powers of the shift.

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